ISRAEL JOURNAL OF MATHEMATICS, Vol. 23, Nos. 3–4, 1976

POLYTOPES WITH CENTRALLY SYMMETRIC FACETS

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ABSTRACT

A new and conceptually simpler proof is given of the theorem of A. D. Aleksandrov and G. C. Shephard, that a d-polytope ($d \geqq 3$), all of whose facets are centrally symmetric, is itself centrally symmetric.

Aleksandrov [1] (for $d = 3$) and Shephard [5] (generally) have proved:

THEOREM. *Let P be a d-polytope* ($d \geqq 3$), *all of whose facets* (($d-1$)-*faces*) *are centrally symmetric. Then P itself is centrally symmetric.*

In outline, Shephard's proof proceeds as follows. Since the image of P under orthogonal projection on to any hyperplane can be dissected into centrally symmetric $(d-1)$-polytopes (namely, the images of the facets of P, in two ways), by a result of Minkowski [4], each such image is centrally symmetric. Then P is itself centrally symmetric, by Blaschke-Hessenberg [2].

We shall give here a new proof of this theorem, which uses only the uniqueness theorem of Minkowski [4], which states that, up to translation, there is at most one polytope whose facets have given outer normal vectors and areas. This theorem is involved in proving the other result of Minkowski just quoted. Since we avoid using the result of Blaschke and Hessenberg, it is clear that our proof is therefore conceptually shorter.

We may suppose P to lie in d-dimensional euclidean space E^d. For each (non-zero) vector $u \in E^d$, let $F(u)$ denote the face of P in direction u; that is, the intersection of P with its support hyperplane with outer normal vector u. If u is such that $F(u)$ is a facet, we shall show that $F(-u)$ is a translate of $F(u)$, so that Minkowski's uniqueness theorem, applied to P and $-P$, shows that P is a translate of $-P$, and so is centrally symmetric.

Received October 3, 1975

To prove this, let F be any $(d-2)$-face of P in $F(u)$. Then there is a facet $F(v)$, such that $F = F(u) \cap F(v)$. It is easily seen that the image of P under the orthogonal projection on to the plane L spanned by u and v is a polygon, whose edges correspond to the facets of P with normal vectors in L, and whose vertices correspond to the $(d-2)$-faces of P which are translates (alternately) of F or $-F$. (This is a special case of part of lemma 2 of Shephard [5].) Thus $F(-u)$ contains a translate of F or $-F$. But this is true for each $(d-2)$-face of P in $F(u)$. Hence $F(-u)$ is a facet, whose pairs of opposite $(d-2)$-faces are just translates of the corresponding parallel opposite $(d-2)$-faces of $F(u)$. By Minkowski's uniqueness theorem again, $F(-u)$ is a translate of $F(u)$, and so the theorem is established.

As Shephard [5] remarks, there easily follows by induction:

COROLLARY. *Let $2 \leqq j \leqq d-1$, and let P be a d-polytope, all of whose j-faces are centrally symmetric. Then for each $j \leqq k \leqq d$, all the k-faces of P are centrally symmetric.*

However, McMullen [3] has shown that, if the condition of the corollary holds with $2 \leqq j \leqq d-2$, then all the faces of P of every dimension are centrally symmetric, so that P is a zonotope (the vector sum of line segments). In such a case, therefore, the theorem of Aleksandrov and Shephard considerably understates the true situation.

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